A proof of Calibration via Blackwell's Approachability Theorem *

Dean P. Foster†

Department of Statistics

University of Pennsylvania

Philadelphia, Pennsylvania 19104-6302

1999

Abstract

Over the past few years many proofs of the existence of calibration have been discovered. Each of the following provides a different algorithm and proof of convergence: Foster and Vohra (1991, 1998), Hart (1995), Fudenberg and Levine (1999), Hart and Mas-Colell (1997)).

*GEB, 1999, 73 - 79

†This work was done while I was visiting the Center for for Mathematical Studies in Economics and Management Science, Northwestern University. Permanent affiliation: Dept. of Statistics, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104. Email:foster@stat.wharton.upenn.edu. I would like to thank Sergiu Hart who provided the proof of the only result in the paper. The algorithm is a modification of the original algorithm in Foster and Vohra (1991).

Does the literature really need one more? Probably not. But this algorithm has two virtues. First it only randomizes between two forecasts that are very close to each other (either p and $p + \epsilon$). In other words, the randomization only hides the last digit of the forecast. Its second virtue is that it follows directly from Blackwell's approachability theorem which shorten the proof extensively. *Journal of Economic Literature* Classification Numbers: C70, C73, C53.

Key words: Individual sequences, worst-case data, regret, learning.

A reasonable definition of calibration can be gotten from the way it is measured by psychologist. They proceed as follows: Ask a subject to make a sequence of forecasts about a sequence of event (for example, whether the subject will win the current hand of bridge she is playing.) Bin these forecasts into 10 bins: $[0, .1]$, $[.1, .2]$, ..., $[.9, 1.0]$. Compute the frequency of of the event occurring within each of these bins. Now plot this empirical frequency of winning the hand vs. mid-points of these 10 bins. If the plot is an approximate 45 degree line, the subject is calibrated. The extent it falls away from the $y = x$ line measures how uncalibrated the subject is.

One curious property of calibration is that there are algorithms that can guarantee calibration even if the algorithm is forecasting the behavior of an opponent. Many proofs of this fact have been given: Foster and Vohra (1991, 1998), Hart (1995), Fudenberg and Levine (1999), Hart and Mas-Colell (1997). This paper will present another such algorithm.

Suppose that the sequence of events being forecast is $X_t \in \{0, 1\}$ for $t = 1, 2, 3 \ldots$. Just before time t a forecast, f_t, is made of the probability that $X_t = 1$. I will assume that this forecast takes on values that are the midpoint of one of the following intervals

$$[0, \tfrac{1}{m}], [\tfrac{1}{m}, \tfrac{2}{m}], \ldots, [\tfrac{i-1}{m}, \tfrac{i}{m}], \ldots, [\tfrac{m-1}{m}, 1].$$

Call the mid-point of interval i:

$$M(i) = \frac{2i - 1}{2m}$$

We can now define the empirical frequency ρ^i as:

$$\rho^i = \begin{cases} \dfrac{\sum_{t=1}^{T} X_t I_{f_t=M(i)}}{\sum_{t=1}^{T} I_{f_t=M(i)}} & \text{if } \sum_{t=1}^{T} I_{f_t=M(i)} > 0 \\ M(i) & \text{otherwise} \end{cases}$$

where

$$I_{f_t=M(i)} = \begin{cases} 1 & \text{if } f_t = M(i) \\ 0 & \text{otherwise} \end{cases}$$

Notice that this is just the weighted distance to the 45 degree line mentioned in the first paragraph. Hopefully, ρ^i lies in the interval $[\frac{i-1}{m}, \frac{i}{m}]$ for all i. If so, the forecast is approximately calibrated. More precisely we can define the L_1 calibration as

$$C_1 \equiv \sum_{i=0}^{m} |\rho^i - M(i)| \overline{I}^i.$$

where

$$\overline{I}^i = \sum_{t=1}^{T} \frac{I_{f_t=M(i)}}{T}.$$

My goal is to come up with an algorithm that will guarantee the L_1 calibration score will converge to zero as time goes to infinity.

If ρ^i does not lie in the interval $[\frac{i-1}{m}, \frac{i}{m}]$, I will measure how far outside the interval it is by two distances: $\overline{d}_t^i$ and $\overline{e}_t^i$ (for deficit and excess) which are defined as:

$$\begin{aligned} \overline{d}^i &= (\tfrac{i-1}{m} - \rho^i)\overline{I}^i \\ \overline{e}^i &= (\rho^i - \tfrac{i}{m})\overline{I}^i \end{aligned}$$

If $\overline{d}^i \leq 0$ and $\overline{e}^i \leq 0$ then ρ^i lies in the ith interval. From these definitions, we can come up with an equivalent definition of the L_1 calibration score:

$$C_1 = \frac{1}{2m} + \sum_{i=1}^{m} \max(\overline{d}^i, \overline{e}^i)$$

Showing that all the $\overline{e}^i$ and $\overline{d}^i$ converge to zero, implies that C_1 converges to $\frac{1}{2m}$.

I will show that the following forecasting rule will drive both of these distances to zero:[1]

1. If there exist an i^* such that $\rho^{i^*} \in [\frac{i^*-1}{m}, \frac{i^*}{m}]$ (in other words, $\overline{e}^{i^*} \leq 0$ and $\overline{d}^{i^*} \leq 0$) then forecast $M(i^*)$.

2. Otherwise, find an i^* such that $\overline{d}^{i^*} > 0$ and $\overline{e}^{(i^*-1)} > 0$ then randomly forecast either $M(i^*)$ or $M(i^*+1)$ with probabilities:

$$\begin{aligned} P(f_{T+1} = M(i^*)) &= \frac{\overline{d}^{i^*}}{\overline{d}^{i^*} + \overline{e}^{(i^*-1)}} \\ P(f_{T+1} = M(i^*+1)) &= \frac{\overline{e}^{(i^*-1)}}{\overline{d}^{i^*} + \overline{e}^{(i^*-1)}} \end{aligned}$$

The following argument shows that it is always possible to find such an i^*. First note that $\overline{d}^1 \leq 0$. So if $e^1 \leq 0$, we have found $i^* = 1$ using case 1. If $\overline{e}^1 > 0$, then consider case 2 with $i^* = 2$. It is satisfied if $\overline{d}^2 > 0$. But if $\overline{d}^2 \leq 0$ it fails. So from the fact that $\overline{d}^1 \leq 0$ this has shown that either $\overline{d}^2 \leq 0$ or we have found an i^* satisfying case 1 or case 2. We can proceed inductively until either we find an i^* which works, or until we have shown that $\overline{d}^m \leq 0$. But, $e^m \leq 0$, so we satisfy step 1 with $i^* = m$.

[1] This algorithm is almost identical to the first algorithm we came up with in 1991 (Foster and Vohra). Unfortunately, the proof that that algorithm worked was very long.

Theorem 1 (Foster and Vohra) *For all $\epsilon > 0$, there exists a forecasting method which is calibrated in the sense that $C_1 < \epsilon$ if T is sufficiently large. In particular the above algorithm will achieve this goal if $m \geq \frac{1}{\epsilon}$.*

I will prove this theorem using Blackwell's approachability theorem. So I will first translate the above algorithm into his setup. First I will describe Blackwell's approachability game.

Consider a two player game, where the first player has m pure strategies and the second player has n pure strategies. The payoff instead of being a single number is a vector, $c(i, j)$. This game will be played repeatedly. Call the average $\bar{c}_T$. There is a convex target set called C. The goal of the first player is to have $\bar{c}_T$ approach this set, whereas the goal of the second player is to keep the first player away from this set. Since it is possible for a series to sometimes be inside C and sometimes to be away from it, it isn't obvious that there has to be a winner of this game. But Blackwell not only showed there is a winner, but how to determine which player wins.

We will have the first player (the approacher) be the statistician and the second player (the excluder) be nature. At each period, the statistician picks i from $1, 2, 3, \ldots, m$ corresponding to the forecasting $M(i)$. Nature picks a value j corresponding to the outcome $\{0, 1\}$. Define

$$\begin{aligned} d^k(i,j) &= (\tfrac{k-1}{m} - j)I_{k=i} \\ e^k(i,j) &= (j - \tfrac{k}{m})I_{k=i} \end{aligned}$$

for $k = 1, 2, \ldots, m$. Then we can treat the vector of both the e's and the d's as the vector valued payoff $c(f, x) \in \Re^{2m}$. In other words:

$$c(i,j) = [d^1(i,j), e^1(i,j), d^2(i,j), e^2(i,j), \ldots, d^m(i,j), e^m(i,j)]$$

which looks in general like:

$$c(i,j) = [0,0,0,\ldots,d^i(i,j),e^i(i,j),\ldots,0,0,0]$$

which can also be written as:

$$c(i,j) = [0,0,0,\ldots,c^{2i-1},c^{2i},\ldots,0,0,0]$$

The goal of the statistician is to drive all of the $\overline{d}$'s and $\overline{e}$'s to be as small as possible. This suggests taking the target set as the all negative orthant:

$$C = \{x \in \Re^{2m} | (\forall k) x_k \leq 0\}$$

Obviously, C is convex. So we have described our statistics problem in terms of an approachability game.

Using Blackwell's theorem, we will see that the statistician can approach the set C. Further, we will see that the algorithm given is exactly the winning strategy given by Blackwell. This means that the statistician can force all the $\overline{d}^i$'s and $\overline{e}^s$'s to be arbitrarily close to zero. Thus the L_1 calibration score will be driven to $1/m$ or less.

Theorem 2 (Blackwell 1956) *Consider the setup given above. The set C is approachable if and only if for all a, there exist a weight vector $w(i)$ such that for all j,*

$$\sum_i w(i)(c(i,j)-b)'(a-b) \leq 0. \tag{1}$$

where b is the closest point to a in the set C (i.e. $b = \arg\min_{x\in C} |x-a|^2$).

The weight vector w in our setting is the vector of probability of forecasting $M(i)$. In other words, $w^k = P(f_t = M(k))$.

An optimal policy for the approacher is to play the weight vector from equation (1) when the current average of the losses is a. To prove Theorem 1 I need to show two things, first that such a w exists, and second that it corresponds to the probabilities chosen by the algorithm. Both of these follow if the probabilities from the algorithm satisfy (1).

The closest point in C to $a \in \Re^{2m}$ is:

$$b = [(a^1)^-, (a^2)^-, \ldots, (a^{2m})^-];$$

where we have defined the positive and negative parts as $x^+ = \max(0, x)$ and $x^- = \min(0, x)$.

Notice that $a - b$ is:

$$a - b = [(a^1)^+, (a^2)^+, \ldots, (a^{2m})^+];$$

Using the property that $(x)^+(x)^- = 0$ we see that $b'(a - b) = 0$. Thus,

$$(c(i,j) - b)'(a - b) = d^i(i,j)(a^{2i-1})^+ + e^i(i,j)(a^{2i})^+$$

So Blackwell's equation is equivalent to:

$$\sum_i w(i) \left(d^i(i,j)(a^{2i-1})^+ + e^i(i,j)(a^{2i})^+ \right) \leq 0 \tag{2}$$

I don't have to actually show that equation (2) is satisfied for all vectors, but merely for all vectors in some set that that contains all the possible payoff vectors. Vectors outside of this set easily satisfy equation (1). The set I will use is $a^{2i-1} + a^{2i} \leq 0$, and $a^1 \leq 0$ and $a^{2m} \leq 0$. Clearly every possible payoff $c(i,j)$ is contained in this set. Recall that these conditions are enough to insure that there exists an i^* that satisfies the algorithm.

If $a^{2i-1} \leq 0$ and $a^{2i} \leq 0$, then the algorithm will assign probability 1 to playing i. So, $\sum w(i)(c(i,j) - b)'(a - b) = 0$, which shows equation (2).

When case (2) of the algorithm is used $a^{2i-1} > 0$ and $a^{2i-2} > 0$, so $a^{2i-3} < 0$ and $a^{2i} < 0$. These follow from the constraints given above. Thus, I want to show

$$\begin{aligned} w(i-1)\left(d^{i-1}(i-1,j)(a^{2i-3})^+ + e^{i-1}(i-1,j)(a^{2i-2})^+\right) &+ \\ w(i)\left(d^{i}(i,j)(a^{2i-1})^+ + e^{i}(i,j)(a^{2i})^+\right) &\leq 0 \end{aligned}$$

But, $(a^{2i-3})^+ = 0$ and $(a^{2i})^+ = 0$, so it suffices to show

$$w(i)d^{i}(i,j)(a^{2i-1})^+ + w(i-1)e^{i-1}(i-1,j)(a^{2i-2})^+ \leq 0$$

or

$$d^{i}(i,j)\left(w(i)(a^{2i-1})^+ - w(i-1)(a^{2i-2})^+\right) \leq 0$$

Since, $d^{i}(i,j) = -e^{i-1}(i-1,j)$. But, from the way that the weights were chosen, the term inside the parentheses is in exactly zero. So we have shown equation (1) holds and hence Theorem 1 holds.

REFERENCES

Blackwell, D. (1956). "An analog of the minimax theorem for vector payoffs." *Pacific Journal of Mathematics*, **6**, 1-8.

Foster, D. and Vohra, R. (1991). "Asymptotic Calibration," technical report, Graduate School of Business, University of Chicago, Chicago IL.

Foster, D. and Vohra, R. (1998). "Asymptotic Calibration," *Biometrika* **85**, 379 - 390.

Fudenberg, D. and Levine, D. (1999). "An easier way to calibrate," *GEB*, 1999 (forthcoming).

Hart, S., (1995) personal communication.

Hart, S. and Mas-Colell, A. (1997). "A simple adaptive procedure leading to correlated equilibrium." technical report, Hebrew University, Center for Rationality.