

Ponzironi Returns: How to distinguish a “con” from a good investment using only statistics

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1 Introduction

Many people believe so highly in their own personally intelligence that they believe they can out-smart everyone else who trades on the New York Stock Exchange (NYSE). This is a common enough belief, that there are whole industries dedicated to helping “lazy” people out smart the market. This industry consists of passing along tips and other investment advice. Thus a large fraction of the advice that float around financial circles is of little use in trading. This means that the problem of seperating good investment possibilities from bad ones is a difficult statistical question.

This paper provides a way of making this decision. First we will discuss what a good investment actually is. There are a multitude of definitions of what a good investment should be. Thus, we will have to first go back to the source and find out what question an investor really is asking. This turns out to be a choice between the following two options:

null: Leave my portfolio of investments as they are.

alternative: Move a small fraction of my portfolio into a new scheme S .

When phrased in this way, we see that the defintion of a good investment will depend on the current holdings of the investor.

One null model of interest is the “buy the market” portfolio. This consists of investing in all the stocks on the NYSE in proportion to their total market value. This value-weighted portfolio has a special place in the hearts of many rabid efficient market enthusiasts. For example, the CAPM (Capital Asset

Pricing Model) recommends that holding the market is the optimal behavior for *all* investors. In other words, it says that no investment can be added to the market portfolio that will improve it no matter what your utility function looks like. Thus if we can find a investment that should in fact be added to the market we have falsified the CAPM. We believe that this is the only correct way to test the CAPM.

The actual test we will construct consists of creating a betting rule. The betting rule will have the property that it isn't allowed to go negative. Thus, if an investment scheme starts out with a single dollar and grows it to a large quantity, it is either very lucky, or it in fact is an investment that we should add to our portfolio. Since this test involves simply placing "bets" we are not making any assumptions about the distribution of returns. In other words, all we are assuming is that the price is right.

We will show that there are investment schemes that will artificially generate good track records. These schemes will fool many common statistical tests into believing that the new investment has beaten chance. But, they fail in fooling our test that is based on betting.

2 Alpha: the parameter of interest

Suppose a statistician is considering adding a new stock S (or a new investment scheme) to her current portfolio of stocks C . Her question is whether this investment scheme will increase her utility or not. Rather than delve into the intricacies of finance, she wants to make this decision based on statistical evidence alone. Let $R_{C,i}$ be the return on her current potfolio at time i , namely $R_{C,i} \equiv C_{i+1}/C_i$. Likewise let $R_{S,i}$ be the return on the candidate new stock to be added to her portfolio. We will make a traditional finance assumption that be can borrow money (or lend it) at the same rata-called the risk free rate: $R_{F,i}$. We will show that if:

$$(R_{S,i} - R_{F,i}) = \alpha_S + \beta_{S,i}(R_{C,i} - R_{F,i}) + \epsilon_i \quad (1)$$

where ϵ_i is a martingale difference array, then she should put at least a small amount of money into stock S if and only if $\alpha_S > 0$. In particular, assume that we can approximate her utility as:

$$U_t(x + EW_t) = x - \kappa x^2. \quad (2)$$

Then we if she considers adding $\epsilon(R_{S,i} - R_{F,i})$ to her portfolio her expected utility as a function of ϵ can be written as:

$$f(\epsilon) \equiv E(U_t(W_t + \epsilon(R_{S,i} - R_{F,i}))) \quad (3)$$

A simple calculation¹ shows that the first derivative of $f()$ is

$$f'(\epsilon) = EX - 2\kappa Cov(W_t, X)$$

Another calculation² shows that

$$\kappa = \frac{E(R_{W,t} - R_{F,t})}{2Var(R_{W,t})}$$

Thus,

$$\begin{aligned} f'(0) &= E(R_{S,t} - R_{F,t}) - \frac{E(R_{W,t} - R_{F,t})Cov(R_{S,t}, R_{W,t})}{2Var(R_{W,t})} \\ &= E(R_{S,t} - R_{F,t}) - \beta E(R_{W,t} - R_{F,t}) \\ &= \alpha_S. \end{aligned}$$

So if $\alpha > 0$ her expected utility will be greater if she invests an positive fraction of her wealth in S then if she leaves it all in her current holdings.

Thus her job is to determine from historical data if α_S is greater than zero or not.

So our investor needs to test

Null hypothesis (H_0): $\alpha_S = 0$,

Alternative hypothesis (H_1): $\alpha_S > 0$.

1

$$\begin{aligned} f'(\epsilon) &= E(U'_t(W_t)X) \\ &= E((1 - 2\kappa(W_t - EW_t))X) \\ &= EX - 2\kappa Cov(W_t, X) \end{aligned}$$

²If we take $X_t = R_{W,t} - R_{F,t}$ then we know that our investor does not desire to purchase nor sell any shares of X_t . This is because her current portfolio is optimal with respects to W . So, $f'(X_t) = 0$. Hence, $E(R_{W,t} - R_{F,t}) = 2\kappa Var(R_{W,t})$

If she can prove that the null is wrong, she has found a good investment. This problem is the basic problem of investing. Unfortunately, she can't believe in a benign nature generating IID normal random variables for her to examine. The reason is that there are sharks out there that will construct investments that will violate any assumptions she adds besides that implied by equation (1). Thus she has to be robust against many more things than a typical statistician worries about. We will show that ignoring these robustness issues will lead her becoming shark bait.

2.1 Why is α_S the right criterion?

This section will show that stock S is desirable iff α_S is greater than zero. To do this will entail several assumptions which will be introduced as we proceed.

Our first assumption is that our investor has a convex/concave? utility which is only a function of wealth. If this isn't the case, things like "hedging" would need to be considered. Second we will assume that this utility can be computed at each point in time and that it is smooth enough to have three derivatives.

Under these two assumptions, we can compute her utility at time $t \approx 0$ given her expected wealth at time t , EW_t .

$$U_t(x + EW_t) = a_t + b_t x + c_t x^2 + o(x^2)$$

By subtracting off a_t and dividing by b_t we can get an equivalent utility function of:

$$U_t(x + EW_t) = x - \kappa_t x^2 + o(x^2)$$

The $\kappa_t = -c_t/b_t$ is called her level of risk aversion. Investors who prefer cash have large κ_t and investors who lost their shirts in dot-bombs probably have lower κ_t .

Now consider the investment decision of adding ϵ of the investment

$$\mathcal{X}_i \equiv (R_{S,i} - R_{F,i}) - \beta_S(R_{C,i} - R_{F,i})$$

to her portfolio. \mathcal{X} is called the excess return. This investment is "self-financing" in the sense that the cost of borrowing the money to pay for it is already included in the returns. Now by the way that β_S is constructed, this return is uncorrelated with her current holdings. Thus,

$$E[U_t(W_t + \epsilon \mathcal{X}_t)] = E[W_t - EW_t + \epsilon \mathcal{X}_t] - \kappa_t E[(W_t - EW_t + \epsilon \mathcal{X}_t)^2] + o(?)$$

Using linearity of expectation we get

$$\begin{aligned}
 E[U_t(W_t + \epsilon \mathcal{X}_t)] &= E[W_t - EW_t] + \epsilon E[\mathcal{X}_t] \\
 &\quad - \kappa_t E[(W_t - EW_t)^2] - 2\kappa_t E[(W_t - EW_t)(\epsilon \mathcal{X}_t)] - \kappa_t \epsilon^2 E[(\mathcal{X}_t)^2] \\
 &\quad + o(?)
 \end{aligned}$$

Which can be simplified by identifying her expected utility under the assumption that ϵ is zero and the fact that \mathcal{X} and $W_t - EW_t$ are uncorrelated:

$$\begin{aligned}
 E[U_t(W_t + \epsilon \mathcal{X}_t)] &= E[U_t(W_t)] + \\
 &\quad \epsilon E[\mathcal{X}_t] - 2\kappa_t \epsilon E[W_t - EW_t] E[\mathcal{X}_t] - \kappa_t \epsilon^2 E[(\mathcal{X}_t)^2] + \\
 &\quad o(?)
 \end{aligned}$$

Thus, if we take t small enough and ϵ small enough, our investor will want to buy at least a small amount of S iff $E[\mathcal{X}_t]$ is greater than zero.

2.2 CAPM

The above argument is one half of what is called the CAPM story. The other half consists of an equilibrium argument. The story proceeds as follows. Suppose two people A and B hold different portfolios that generate excess returns R_A and R_B . If the correlation between these two is less than perfect, then either the regression of R_A on R_B has a non-zero intercept, or visa versa. Thus, at least one of these two investors is behaving non-optimally.

So if take as an assumption that all investors have made optimally investments, then we see that all of their returns must be perfectly correlated. Call an investment that generates this return “the market.” Suppose there existed an investment that had a non-zero α when regressed against the market. Then if $\alpha > 0$, everyone in the entire model would want to buy some of this investment. Since there are no sellers, the price would go up until an equilibrium was reached. This would occur at exactly the point where $\alpha = 0$. Thus, by adding the assumption of optimal investments and that the system is in equilibrium is enough to show that there is one investment that everyone holds in common.

Why is this investment called “the market”? Since everyone is generating the exact same return series, we can assume that they have equalized their investments between themselves so that they all hold the exact same mixture of stocks. Since this now adds up to exactly the total holdings of all the investors, it is in fact the entire market. So the name makes sense.

Of course, the theory of testing presented in this paper doesn't require the CAPM to hold—but it does provide the correct test of whether the CAPM is correct or not. The CAPM says that it is impossible to find an asset that has $\alpha \neq 0$ when regressed on the Market. That is all it says. Many other people have read other things into the CAPM by adding other assumptions, but the problem with doing that is that a rejection of the null no longer directly address the CAPM but also the added assumptions.

3 The basic rule

We can state the hypothesis test as:

Null: M_T is a martingale.

Alternative: M_t is a sub-martingale

where the differences $\Delta M_t = M_t - M_{t-1}$ are defined as:

$$\Delta M_T \equiv \sum_{i=1}^T (R_{S,i} - R_{F,i}) - \beta_S (R_{C,i} - R_{F,i}). \quad (4)$$

Suppose that we are looking at a sequence of ΔM_i 's which are supposed to form a martingale difference array. In other words, $E(\Delta M_i | \mathcal{F}_{i-1}) = 0$ for all i , where \mathcal{F}_{i-1} represents all possible knowledge at time $i-1$. But, do they have mean zero or not?

If we assumed that the ΔM_i 's were independent normal random variables, then we would know that t-test:

$$t = \frac{\sum_{i=1}^N \Delta M_i}{\sqrt{\sum_{i=1}^N (\Delta M_i)^2}} \quad (5)$$

would be the optimal test statistic. But, this assumption of IID normality is not tenable. In fact, we should be unwilling to even assume IID. So what test statistic can we use that will give us reliable results without making these assumptions?

Zhao (citation?) considered many ways of solving this problem. We will use the following solution.

Assumption 1 *There exists a constant B such that for all i we have*

$$\Delta M_i \geq -B \text{ almost surely.}$$

Our test statistic will be:

$$\widetilde{M}_\lambda = \prod_{i=1}^N (1 + \lambda \Delta M_i) \quad (6)$$

If we have chosen $\lambda \geq 1/B$ for the B in Assumption 1, then $E(\widetilde{M}_\lambda) = 1$ and $\widetilde{M}_\lambda \geq 0$. Thus, using Markov's inequality, we see that $P(\widetilde{M}_\lambda \geq k) \leq 1/k$.

3.1 How much do we give up in the robust test?

We will show that using the robust test (6) instead of the t-test (5) will not give up as much as we might expect.

First we need to find the optimum value for λ . We will assume that $\lambda B \geq -.8$. If this is true, then (via Mathematica) we see that

$$\log(1 + \lambda \Delta M_i) \geq \lambda \Delta M_i - (\lambda \Delta M_i)^2/2 - |\lambda \Delta M_i|^3.$$

and

$$\log(1 + \lambda \Delta M_i) \leq \lambda \Delta M_i - (\lambda \Delta M_i)^2/2 + |\lambda \Delta M_i|^3.$$

So,

$$\begin{aligned} \widetilde{M}_\lambda &= \prod_{i=1}^N (1 + \lambda \Delta M_i) \\ &= \exp\left\{\sum_{i=1}^N \log(1 + \lambda \Delta M_i)\right\} \\ &\geq \exp\left\{\sum_{i=1}^N \lambda \Delta M_i - (\lambda \Delta M_i)^2/2 - |\lambda \Delta M_i|^3\right\} \end{aligned}$$

If we now let

$$Z = \frac{\sum_{i=1}^N \Delta M_i}{\sqrt{\sum_{i=1}^N \Delta M_i^2}}$$

and,

$$C = \frac{\sum_{i=1}^N |\Delta M_i|^3}{(\sum_{i=1}^N \Delta M_i^2)^{3/2}}$$

where C is the cubic moment. Typically it should be of size $O_P(1/\sqrt{N})$ if there aren't many extreme values. We could also use $\max(\Delta M_i^3, 0)$ instead

of this $|\Delta M_i|^3$ in the definition of this third moment. This might make a difference in actual applications—but doesn't make much difference theoretically.

Finally define a modification of λ as:

$$\kappa = \lambda \sqrt{\sum_{i=1}^N \Delta M_i^2}$$

Now we can write our approximation for M as:

$$\begin{aligned} \log(\widetilde{M}_\lambda) &\geq \lambda \sum_{i=1}^N \Delta M_i - \lambda^2 \sum_{i=1}^n \Delta M_i / 2 - \lambda^3 \sum_{i=1}^n |\Delta M_i|^3 \\ &\geq \kappa Z - \kappa^2 / 2 - \kappa^3 C \end{aligned}$$

Which, if we ignore the $\kappa^3 C$ term, obtains a maximum value of $Z^2/2$ when $\kappa = Z$. Thus the “p-value” of \widetilde{M}_λ is approximately:

$$\text{p-value} = 1/\widetilde{M}_\lambda \approx e^{-Z^2/2}$$

This is reasonably close to the p-value for the IID normal of

$$\text{p-value} \approx 1 - \Phi(Z) = \frac{e^{-Z^2/2}}{\sqrt{2\pi Z}}.$$

The above approximation requires that we can figure out λ in advance. Note that $\lambda = \sum_{i=1}^N \Delta M_i / \sum_{i=1}^N \Delta M_i^2$. We can reasonably guess the value of $\sum_{i=1}^N \Delta M_i^2$ since the “variance” is observable under the assumption that the ΔM_i 's are all in fact small. If we want to perform a test which has size α^* , then we can back out what Z should be to make the \widetilde{M}_λ test have the desired size: namely $Z = \sqrt{2 \log(1/\alpha^*)}$. Thus,

$$\lambda = \frac{\sqrt{2 \log(1/\alpha^*)}}{\sqrt{\sum_{i=1}^N \Delta M_i^2}} \tag{7}$$

where α^* is the desired size of the test.

3.2 More than one stock

This methodology extends nicely to considering the case of more than one stock. All we need do is spread a dollar initial investment over all of the stocks we want to consider and see if our final wealth is greater than \$20. If so, we can reject at the .05 level the hypothesis that *none* of the investments are valuable.

Notice that the Z required for the robust test and the Z required for using Bonferonni are almost identical. The wonder of this is that it works both directions. There are ways to get very close to the bonferonni p-value bound and hence make it tight. These methods will thus show that the robust methodology is also approximately tight. So again, we haven't given up very much by dropping the IID and normality assumptions.

4 A variety of ways to losing money

We will start by discussing a variety of ways that an investor can be con'ed into losing money. The point of this section is to show if α_S is truly zero, there are still many ways of getting returns that look impressively better zero.

4.1 Ponzi schemes and Bonferonni

4.2 What is a Ponzi scheme?

1. Sometimes called pyramid schemes, multi-level marketing schemes, air-plane schemes, sometimes simply mail-fraud
2. Start with 100 investors. All give 100 dollars.
3. Each month find 10 new investors to each give 100 dollars.
4. Pay this new money to the existing investors
5. Keep finding new investors each month.
6. Once you can't find anyone else new the scheme is said to crash
7. Our goal: fake a Ponzi scheme (legally?)

4.3 Financial Ponzi schemes

Story 1 (PBS's Mathnet (modelled after dragnet)) 1. send out 64 emails: 32 forecasting market up 32 forecasting market down

2. send out 32 emails to those who we got right in first rund: 16 up 16 down

3. send out 16 emails: 8 up 8 down

4. send out 8 emails: 4 up 3 down

5. send out 4 emails: 2 up 2 down

6. send out 2 emails: 1 up 1 down

7. Now send out email asking for money for next forecast

Story 2 (Many hedge funds) 1. Start 64 hedge funds: 32 leverage market up, 32 leverage market fall

2. Fold losing 32

3. Of 32 winners: 16 leverage market up, 16 leverage market fall

4. Fold losing 16

5. ...

6. Last 6 month track record: doubling every month

7. Show track record to very rich person—ask for one million dollar investment

Story 3 (Searching historical returns) 1. Start with 64 rules for picking a stock (think leveraged)

2. Each month or two kill off the bottom 1/2 of the rules

3. After a year or so, you have one rule that has grown spictularilly

4.3.1 Cure: Bonferonni

How big might the t-statistic be following such a scheme?

1. Suppose you do n different tests
2. Bonferonni p-value = $n * \text{regular p-value}$
3. If JMP won't do it for you, use $\sqrt{2 \log(n)}$ for significance
4. If you require Bonferonni significance, you will rarely fall into the trap of 3rd story.

4.4 Faking a good empirical track record

1. Would a scheme that returned 2years impress you?
 - (a) last 10 years?
 - (b) last 20 years?
 - (c) last 100 years?
2. Our goal is to generate such impressive returns without having to understand actual stocks and bonds.
3. A Ponzi scheme for quiet customers
 - (a) Start with 1000 customers who each give 1000 dollars
 - (b) Each month, tell 10bust and no longer has any money in it.
 - (c) Distribute the money of those busted 10players.
 - (d) After several rounds, you have a "client list" of customers each whom has seen 10several rounds.
 - (e) After 25 periods you are down to 10 people. The scheme basically crashes at this point. Notice that these people have seen a 10hold 100,000 dollars!
4. Problem: The busted part of your client list will complain and the scheme will be busted by the police.
5. Principle: Dead people don't talk.

6. Better principle: Non-existent people talk even less!
7. A Ponzirone scheme (pick one random player from your original list and track their wealth.)
 - (a) Start with 1 customer who gives 1000 dollars
 - (b) Each month, place a bet which has a 9/10 chance of winning the whole thing.
 - (c) If lucky, give money to client
 - (d) If unlucky, tell client that they are busted
 - (e) You have a 1/100 chance of making as far as 25 rounds before your one client crashes.
 - (f) After 25 periods you are down to 10 people. The scheme basically crashes at this point.

4.4.1 Estimating the probability of rare events.

1. Suppose you are watching a biathlon (skiing and shooting) and the shooter has hit 25/25 so far. What is her chance of missing?
2. Suppose the true probability of missing is p , then to get 25/25 would have a chance of $(1-p)^{25}$. This is a very small probability IF $p < 3/25$. But, if $p < 3/25$, the chance of this occurring is greater than 5%.
3. Rule for seeing no misses in n shots so far:
 - (a) $0 < \text{chance of missing} < 3/n$ is 95% confidence interval
 - (b) $0 < \text{chance of missing} < 5/n$ is 99.5% confidence interval
 - (c) $0 < \text{chance of missing} < 10/n$ is 99.995% confidence interval
 - (d) $0 < \text{chance of missing} < k/n$ is $1 - \exp(-k)$ confidence interval
4. Example: Suppose you are watching 1000 biathlons in order to possibly pick one to join your team. The best shooter never misses. How should you estimate her chance of missing? Use 1 - .05/1000 for confidence level. That means that $k = 10$.

4.4.2 How good a record can we fake?

1. Above scheme requires the possibility of losing everything.
2. What if the amount we can lose is bounded?
 - (a) Any instrument for which options exist, can be converted to having bounded downside. (Of course this lowers the return.)
 - (b) Options can be "replicated" as long as the instrument is publically traded.
 - (c) So publically traded assest can have their downside protected.
 - (d) (Simple rule: sell if price drops more than 10any month. Your sell might not get executed until it has dropped a bit further. This will depend on market thickness. Then hold cash for the rest of the month.)
 - (e) Hedge funds can't have options written against them since buying and selling only occurs monthly instead of continuously.
3. With bounded loss, we are forced to use a smaller disaster. Hence we can't fake as good performace.

4.4.3 Why read a mutual fund prospectus?

1. An alternative to using options to bound the loss is to read the prospectus.
2. A mutual fund who can use leverage can generate larger swings.
3. If only stocks can be purchased, downsides can be much lower
4. If only big stock can be purchase and they must be diversified, then the downside is lower still.

4.4.4 Buy side: The three disasters rule

1. Determine what a total disaster would be by one of the following methods: (listed best to worst)
 - (a) Use actual options then the downside is legally determined

- (b) Use active trading. Assume you monitor the value of the fund and anytime it drops by X again until next period. Charge yourself Y transaction costs AND how much it might have dropped past X
- (c) Estimate a disaster drop by reading the prospectus. How bad might things go? If the fund is prohibited from purchasing options, then it can't place as big bets as if it is allowed to purchase options. How diversified must the fund stay?
- (d) Use -100

2. Make a column of empirical returns
3. Add three disasters to this column
4. Test if the average return for this column is higher than expected.
5. Add three disasters to the data that you have on the fund. It doesn't matter if you have daily or month or yearly data, you still add three disasters.
6. Test the null hypothesis that the returns on the fund with the three disasters are equal to the returns on the market. (Or compare against cash, or what every is your alternative.)

4.4.5 Example: The market vs. t-bills

Suppose you want to show that the market is a better investment than t-bills. Suppose you are willing to watch the market carefully. In other words you will sell off the market in the middle of any month that drops more than 10%. Now take about 5 to 10 years of monthly data, add three observations of -.10 to the end of the column and test if the returns are higher than the returns for t-bills over this interval of time.

4.5 Risk

1. Pick a partner and do the dice simulation

4.5.1 Debriefing

1. Show of hands: Who liked: Green? Red? White?
 - (a) White is like T-Bills/cash.
 - (b) Green is like the market
 - (c) Red is like an "internet stock."
2. What was your final wealth for white/green/red?
 - (a) Make a histogram of each (0 - 2000 for white, 0 - 20,000 green)
 - (b) Red mostly goes to zero, but someone will prob. be lucky (billions of dollars final wealth)
3. Growth is determined not just by the mean but also by the variance
 - (a) Joe's boss gets mad at him, and cuts his pay 10
 - (b) The next day, his boss says he isn't mad anymore and so gives him a 10
 - (c) Why isn't Joe completely happy about this?
 - (d) Notice: Joe is behind 1 percent from where he started
4. True growth rate = mean - 1/2 variance
 - (a) Sometimes called the log-growth rate
 - (b) Related to a utility function that looks like a log
5. To tame distribution of red, compute the log(final wealth)
 - (a) either use base 10 logs (to make the plotting easier) or base e logs (to make the drift rate meaningful) But make sure that everyone in class is using the same base!
 - (b) Have each group compute their three logs of final wealth
 - (c) Sketch new histograms for each color (or compute logs in JMP)
 - (d) red is much more spread out, but drifts left

6. Work out what should happen for Pink
 - (a) via example, show that the mean of pink is about 1/2 that of red (in other words, the mean of white is about zero)
 - (b) Via example, show that the standard deviation of Pink is about 1/2 that of Red (In other words, the variance of white is almost zero)
 - (c) compute the expected log growth rate for pink (mean - 1/2 variance)
 - (d) If using log-base-e, "guess" where a typical groups final pink wealth should be
 - (e) collect pink data on log scale

7. Moral: Need both the mean and the variance to evaluate a stock / bond / portfolio / option / derivative product.
 1. long run growth rate = mean - variance / 2
 2. $\text{Var}((A + B)/2) = \text{Var}(A)/4 + \text{Var}(B)/4 + \text{Cov}(A,B)/2$
 3. If Cov is approximately zero, $\text{Var}((A + B)/2) = \text{Var}(A)/4 + \text{Var}(B)/4$.
 4. Optimum investment: How much market should you own?
 - (a) Suppose interested in long run growth rate
 - (b) Goal: maximize mean - 1/2 variance
 - (c) If we put w fraction of wealth in market
 - i. mean is .07k
 - ii. SD is (.22k)
 - iii. Variance is $(.22k)^2$
 - (d) goal: maximize $.07k - .22^2 k^2 / 2$
 - (e) answer: take derivative
 5. long run growth rate = mean - variance / 2
 6. optimizing LRGR:

- (a) How leveraged should one be?
- (b) $\text{mean} = \alpha * (\text{mean} - \text{RiskFreeRate})$
- (c) $\text{variance} = \alpha^2 \text{ variance}$
- (d) optimize a quadratic: $-b/2a$
- (e) Market details
 - i. If we put w fraction of wealth in market
 - ii. mean is $.07w$
 - iii. SD is $(.22w)$
 - iv. Variance is $(.22w)^2$
 - v. goal: maximize $.07w - .22^2w^2/2$
 - vi. optimize a quadratic: $b/2a = (-.07)/(-2 * .22^2/2) = 1.44$
 - vii. Slightly leveraged
- (f) optimum investment: $\alpha = (\text{mean} - \text{RiskFreeRate})/\text{variance}$
- (g) Units work out since everything is in returns

7. Look at stock market

- (a) look at whole series (find annual mean and variance)
- (b) look at recent series
- (c) figure out optimal investment
- (d) compute LRGR and do it empirically

4.5.2 Optimal investing

1. What if you have more than one instrument?
 - (a) Good life: means add AND variances add
 - (b) Life is 1/2 good. means always add
 - (c) Variances only add if uncorrelated
2. How do you make something uncorrelated with the market?
 - (a) Look at $\text{residInvestment} = (Y - \text{RF}) - \text{beta}(M - \text{RF})$

- (b) sell beta of (M-R) buy (Y-RF)
 - (c) Claim resid is uncorrelated with market if beta is choosen correct (otherwise residuals wouldn't be flat)
 - (d) So use beta = beta of regression of Y on market
 - (e) mean is now alpha of regression (just like CAPM)
 - (f) SD = SD from regression
3. How much to buy of each? (if uncorrelated)
 - (a) optimize each separately
 - (b) buy correct amount of market
 - (c) buy a α/SD^2
 - (d) Key thing: is alpha significantly bigger than zero?
 4. If you are more risk adverse, buy more of the risk free and less of risky assets

4.6 CAPM

1. Notice, if $\alpha > 0$, everone wants to buy
2. Notice, if $\alpha < 0$, everone wants to sell
3. Hence price is not in equilibrium unless $\alpha = 0$

4.6.1 Putting it all together

1. Figure of merit: $F = \alpha^2/MSE$
2. Growth of log optimal portfolio increases by $F/2$
3. It takes $4/F$ years to statistically prove this investment is profitable using out-of-sample data
 - (a) Find an F of 1, and CAPM is dead in 4 years
 - (b) Find an F of .01 and CAPM is indistinguishable from "better model" for next 400 years

- (c) For reference: T-bills vs. cash has $F = (.03/.01)^2 = 9$ (takes several months to prove better)
- (d) For reference: Market vs. t-bills has $F = (.07/.2)^2 = .12$ (takes 30 years to prove)

4.6.2 Example: The Quant-Jock

1. Suppose you put a quant jock in a cage and ask him questions: Is XYZ corp a good investment? Five minutes later he gives an answer.
2. A game the traders play is guess what the quant jock will say. No one can guess what he will say better than 50/50.
3. To the world, the quant jock looks like a coin toss
4. BUT, over the course of a year the quant jock's "buys" grow by 2 percent a year compared to his "sells".
5. What is his figure of merit?
6. Portfolio: Buy his "buys," short his "sells". Buy portfolio is correlated at least .99 with sell portfolio (same variance). Thus difference as has a variance of $(1 - R^2) * \text{variance of market} = .02 * (.2)^2$.
7. Figure of merit: $(.02)^2 / (.02 * .2^2) = .5$
8. Doubles every two years
9. Takes 8 years to "prove" method is successful

5 Convincing bet

Suppose you have a friend who claims to have a secret scheme for making money at a casino. What would it take to convince you that they have in fact succeeded? If you gave them a dollar, and they came back with 2 dollars and said: "See, I made money at the casino." you would not be impressed. All they had to do was go up to a rollet wheel and bet on black—which pays 2 dollars about 1/2 the time. So they could accomplish this task by simply winning a single coin toss. But, suppose you gave them a dollar and they came back with loads of money, say, M dollars. If in fact they were placing

bets that favored the house, then their expected amount of money at the end of their betting sequence would be less than one dollar. Using Markov's inequality, then the probability of ending up with more than M dollars is less than $1/M$. If M is sufficiently large, you will be impressed—there is no high probability route to generating a large M so you should reject the idea that they are simply placing losing bets (on average) and counting on good luck to save them.

Traditional statistics says that achieving a 5% chance is impressive enough to believe it was generated by something other than chance. This suggests setting M at 20.

Going back to our estimated α we would then say that it is significantly greater than zero if $e^{\alpha t} > 20$, where t is the amount of time we observed for. In other words, if we had invested a small amount in this investment, it would have grown by a factor of 20 over the t periods we observed the data. This suggests our first rule;

excess returns: We have excess returns if α is greater than $\ln(20)/t$.

1. Merton (1987) "Is it reasonable to use the standard t-statistic as a valid measure of significance when the test is conducted on the same data used by many earlier studies whose results influenced the choice of theory to be tested?" (p 107 *Macroeconomics and finance*, R. Dornbusch, S. Fischer, and J. Bossons (eds.), MIT press.